Final exam for Kwantumfysica 1 - 2008-2009
Tuesday 20 January 2009, 13:00 - 16:00

READ THIS FIRST:

- Note that the lower half of this page lists some useful formulas and constants.
- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write clearly the total number of answer sheets that you turn in.
- Note that this exam has 4 questions, it continues on the backside of the papers!
- Start each question (number 1, 2, 3,4) on a new answer sheet.
- The exam is open book within limits. You are allowed to use the book by Liboff, the handout *Extra note on two-level systems and exchange degeneracy for identical particles*, and one A4 sheet with notes, but nothing more than this.
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or "derive", you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.
- If you are ready with the exam, please fill in the **course-evaluation question sheet**. You can keep working on the exam until 16:00, and fill it in shortly after 16:00 if you like.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Fourier relation between x-representation and k-representation of a state

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\Psi}(k) e^{ikx}\, dk$$

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx}\, dx$$

Problem 1

A certain atom is in a state with its total orbital angular momentum vector $\boldsymbol{L}$ (described by the operator $\hat{L}$) defined by orbital quantum number $l = 1$.

a) What is in this case the length of this vector for total angular momentum $\boldsymbol{L}$?

For the system in this state, the operator for the z-component of angular momentum is $\hat{L}_z$. It has three eigenvalues, $+\hbar$ (with corresponding eigenstate $|+_z\rangle$), $0\hbar$ (with eigenstate $|0_z\rangle$), and $-\hbar$ (with eigenstate $|-_z\rangle$). This operator can be represented as a matrix, and the ket-states as column vectors, using the basis spanned by $|+_z\rangle$, $|0_z\rangle$ and $|-_z\rangle$, according to

$$\hat{L}_z \leftrightarrow \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad |+_z\rangle \leftrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |0_z\rangle \leftrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix}, \text{ and } \quad |-_z\rangle \leftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Using this same basis for the representation, the operator and eigenstates for the system's x-component of angular momentum are given by

$$\hat{L}_x \leftrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad |+_x\rangle \leftrightarrow \begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix}, \quad |0_x\rangle \leftrightarrow \begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{pmatrix}, \text{ and } \quad |-_x\rangle \leftrightarrow \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix}.$$

b) *Calculate* with this information what the eigenvalues are that belong to $|+_x\rangle$, $|0_x\rangle$ and $|-_x\rangle$.

c) At some point the system is in the normalized state $|\Psi_1\rangle = \sqrt{\tfrac{1}{8}}\,|+_z\rangle + \sqrt{\tfrac{3}{8}}\,|0_z\rangle + \sqrt{\tfrac{4}{8}}\,|-_z\rangle$. Calculate for this state the expectation value for angular momentum in z-direction and the expectation value for angular momentum in x-direction.

d) At some point the system is in the normalized state $|\Psi_2\rangle = \sqrt{\tfrac{1}{3}}\,|+_z\rangle + \sqrt{\tfrac{1}{3}}\,|0_z\rangle + \sqrt{\tfrac{1}{3}}\,|-_z\rangle$, and you are going to measure the x-component of the system's angular momentum. What are the possible measurement results?

d) continued !

→ Calculate for each possible measurement result the probability given that the system is in state $|\Psi_2\rangle$.

e) At some point the system is in the state $|\Psi_3\rangle = i\,|+_z\rangle + 2\,|0_z\rangle - i\,|-_z\rangle$. Note that this state is not normalized. Calculate for this state $\langle \hat{L}_z \rangle$.

f) At some point the system is in the normalized state $|\Psi_4\rangle = \sqrt{\tfrac{1}{2}}\,|+_z\rangle + \sqrt{\tfrac{1}{2}}\,|-_z\rangle$. Calculate for this state the quantum uncertainty ΔL_z in the *z*-component of the system's angular momentum.

Problem 2

Consider the following quantum system. It has a single quantum particle in a one-dimensional particle-in-a-box system, where the potential for the particle outside the box is infinite, and inside the box the potential $V = 0$. The position of the electron is described by a coordinate x. The width of the box is a, with the walls at $x = -a/2$ and $x = +a/2$. The mass of the particle is m. The system is weakly coupled to a very cold environment, such that it always relaxes to its ground state by spontaneous emission of energy.

a) Write down a valid description for the ground state of this system in x-representation, and give the energy of the ground state E_g (with respect to $V = 0$) in terms of a, m and natural constants.

Now one puts a second quantum particle in this box, which is identical to the first particle. Again, the complete system is relaxing to its ground state.

b) Assume that the particles behave as bosons, such that they can only be in a symmetric state (neglect the spin of the particles, or assume that they have the same spin state). Write down a valid description for the ground state of the complete system in x-representation, and give the energy of the ground state with respect to $V = 0$ (if possible express the answer in terms of E_g of question **a**)).

c) Assume that the two particles behave as fermions, such that they can only be in an antisymmetric state (neglect the spin of the particles, or assume that they have the same spin state). Write down a valid description for the ground state of the complete system in x-representation, and give the energy of the ground state with respect to $V = 0$ (if possible express the answer in terms of E_g of question **a**)).

d) Consider the system and quantum state of question **c**). Show that the probability for a certain measurement result when measuring the position of both particles does not change when exchanging the two particles.

Problem 3

Note: You must use Dirac notation for solving this problem.

Consider a system with a time-independent Hamiltonian $\hat{H}$, that has only two energy eigenstates $|\varphi_1\rangle$ and $|\varphi_2\rangle$. These have, respectively, two different energies E_1 and E_2 with $E_1 = 2.110\cdot10^{-25}$ J and $E_2 = 3.165\cdot10^{-25}$ J.

a) This system has a magnetic dipole moment that is described by the operator $\hat{D}$. This operator $\hat{D}$ commutes with $\hat{H}$. Assume that at some point in time $t = 0$, the system is in some arbitrary superposition of the two energy eigenstates. Show that this system will then never have any oscillations of $\langle \hat{D} \rangle$ in time.

Use Dirac notation and the operator $\hat{U} = e^{\frac{-i\hat{H}t}{\hbar}}$.

b) This system also has an electrical dipole moment that is described by the operator $\hat{A}$. For this system,

$$\langle \varphi_1|\hat{A}|\varphi_1\rangle = 0 \quad , \quad \langle \varphi_2|\hat{A}|\varphi_2\rangle = 0 \quad , \quad \langle \varphi_1|\hat{A}|\varphi_2\rangle = \langle \varphi_2|\hat{A}|\varphi_1\rangle = A_0 \quad (\text{with } A_0 > 0).$$

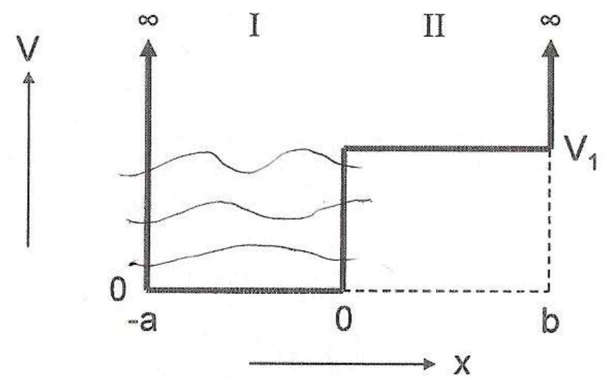
You can prepare the system in the following initial state (at some time defined as $t = 0$) with full control over the phase ϕ,

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}|\varphi_1\rangle + \frac{e^{i\phi}}{\sqrt{2}}|\varphi_2\rangle \quad .$$

What is the value of ϕ that you must use (with $0 < \phi < 2\pi$) if you want that the expectation value of the electrical dipole has a maximum at $t = 1$ ns?

Problem 4

Consider the following one-dimensional potential $V(x)$ for a quantum particle with mass m. The potential V_1 is large enough to have at least 3 energy levels in the system below V_1.



a) This potential looks rather similar to the potentials of the finite and infinite potential well problems. For $b \to \infty$, sketch the wavefunctions of the three energy eigenstates with lowest energies that you expect.

b) Explain by reasoning whether the ground state energy of this potential well is higher or lower than the ground state energy of an infinite potential well of width a.

Questions **c)** - **e)** concern solving the time-independent Schrödinger equation for this system.

c) [Read question d) and e) first] Solve the time-independent Schrödinger equation in the two indicated areas I and II for energies $0 < E < V_1$ and write the obtained wavefunctions in the most general way possible.

d) Reduce the amount of unknown constants in your wavefunctions by (here first only) using the boundary conditions at $x = -a$ and $x = b$.

e) Now also use the boundary condition(s) at $x = 0$. Show that you can now write an equation for finding E of an eigenstate without using any unknown constants (that is, in terms of E, V_1, a, b and m only).